# Some applications of Deep Learning algorithms for PDEs


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# Main topic of the lecture

**Objectives :**

- Present some applications of the *Deep Galerkin* Algorithm and *Deep BSDE* Solver for solving *PDE*.
- Show how both of theses algorithms can be efficiently implemented in Python with PyTorch.

# Contents

We are giving to present some applications of the Deep Galerkin algorithm for solving PDE in a general form :

$$\partial_v(t,x) = \mathcal{H}[v](t,x) \quad (t,x) \in [0,T) \times \mathbb{R}^d \tag{1}$$

$$v(T,x) = g(x), \quad x \in \mathbb{R}^d \tag{2}$$

where $\mathcal{H}$ is an operator which can contain multiples derivatives of $v$ with respect to $x$. Given a smooth function $\omega$ on $[0,T] \times \mathbb{R}^d$ , we define :

$$\mathbb{L}(\omega) = \mathbb{E}[|\omega(T,\mathcal{X}) - g(\mathcal{X})|^2] + \mathbb{E}[|\partial_t\omega(\tau,\mathcal{X}) - \mathcal{H}[\omega](\tau,\mathcal{X})|^2] \tag{3}$$

where $(\tau,\mathcal{X})$ are independant random variables $\sim v_T \otimes v_d$ supported on $[0,T] \times \mathbb{R}^d$.

### Remark

*From the definition of $v$, it is clear that $v$ is a solution to (1) if and only if $v$ achieves the minimum of $\mathbb{L}$. However, optimization problem (3) is infinite dimensional and not feasible numerically.*

The idea is therefore to parametrize the space of smooth functions $\omega$ by the class of neural networks $\mathcal{U}_\theta$ on $[0, T] \times \mathbb{R}^d$ which reduces to a finite dimensional optimization problem :

$$\inf_\theta \mathbb{L}(\mathcal{U}_\theta) \tag{4}$$

**Remark**

- *The optimization problem (4) is done through stochastic gradient descent based on the expectation representation* $\mathbb{L}(\mathcal{U}_\theta) = \mathbb{E}\big[l(\theta, \tau, \mathcal{X})\big]$ *with*

$$l(\theta, t, x) = |\mathcal{U}_\theta(T, x) - g(x)|^2 + |\partial_t \mathcal{U}_\theta(t, x) - \mathcal{H}[\mathcal{U}_\theta](t, x)|^2$$

- *Equation (4) is finite dimensionnal because* $\theta$ *represents the values of the weights and the bias of the NN which are finite dimensionnal and optimized during the training process.*

We will do some experiments of the Deep Galerkin for the $B-S$ in dimension $d=1$ for different type of PDE. Under option pricing theory, for an European option with price at time $t$ denoted by $C(t, S_t)$ we know that we have the following PDE representation for the option price $C$ defined on $[0, T] \times \mathbb{R}_*^+$ as :

$$\begin{aligned} \partial_t C + \mathcal{L}C - rC = 0, \quad (t,x) \in [0, T) \times \mathbb{R}_*^+ \\ C(T,x) = g(x), \quad x \in \mathbb{R}_*^+ \end{aligned} \tag{5}$$

where the infinitemisal generator is given by :

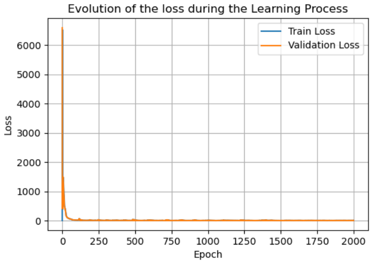
$$\mathcal{L}v(t,x) = rx\partial_x v(t,x) + \frac{1}{2}\sigma^2 x^2 \partial_x^2 v(t,x)$$

**Remark**

*For the numerical experiments, we choose $r = 0.02$, $\sigma = 0.2$ , $T = 1$ and we choose in this special setting $\tau$ and $\mathcal{X}$ to take values in $[0, T(\times(0, 200)$ through a meshgrid. In usual case, people assume that $\tau \sim \mathcal{U}([0, T])$ and $\mathcal{X} \sim \mathcal{U}(D)$ where $D$ is a bounded domain of interest.*

# Numerical results in the $B - S$ setting

## A forward contract



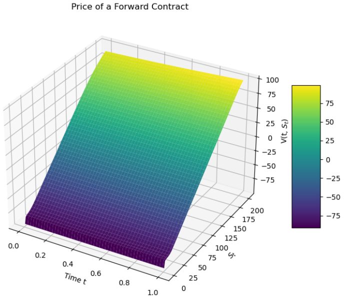




Figure: Evolution of the training and validation losses for the learning process of the PDE (5) related to the price of a forward contract (Case of $g(x) = x - K$ for $K = 100$)

### Remark

*We recover the linear relation between $S_t$ and $C(t, S_t)$ for the case of the forward contract as we know that $C(t, S_t) = S_t - Ke^{-r(T-t)}$.*

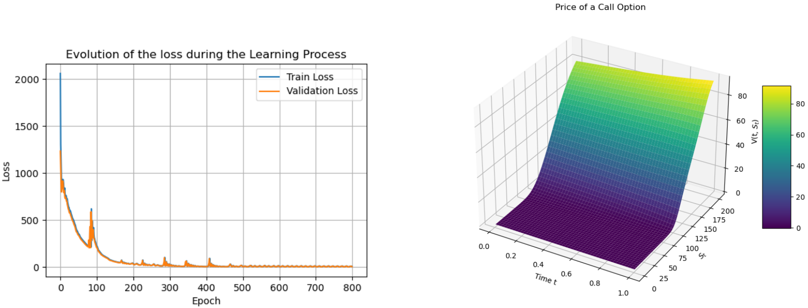



Figure: Evolution of the training and validation losses for the learning process of the PDE related to the price of a call option (Case of $g(x) = (x - K)^+$ for $K = 100$)

### Remark

*We recover the classic form for a call option noticing that $t \rightarrow C(t, S)$ is an increasing function.*

It can be shown that the *Credit Valuation Adjustment* (CVA) in a default intensity model (such that $\mathbb{P}(\tau_C \geqslant t) = e^{-\lambda^C t}$) is solution to the following PDE :

$$\partial_t \phi(t,x) + \mathcal{L}\phi(t,x) - (r+\lambda^C)\phi(t,x) + (1-R^c)(V_t)^+\lambda^C = 0, \quad (t,x) \in [0,T(\times \mathbb{R}_*^+ \tag{6}$$

$$\phi(T,x) = 0, \quad x \in \mathbb{R}_*^+$$

where :

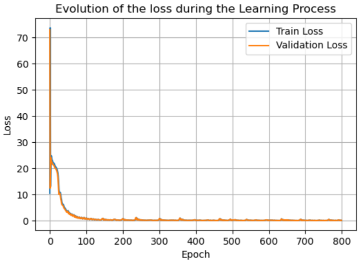
- $\lambda^C$ is the default intensity of the counterparty assumed to be constant.
- $R^C$ is the recovery rate in case of default of the counterparty.
- $(V_t)^+$ is the value of the exposure of the portfolio / derivative involved between both counterparties.

**Remark**

*In our numerical experiments, we will take the portfolio to be a standard call option with same characterics as before but with $K = 110$. Moreover, we will assume $R^C = 0$ and show the results for 2 different values of $\lambda^C$.*

# Numerical results in the $B - S$ setting

## CVA on a Call Option with low $\lambda^C$



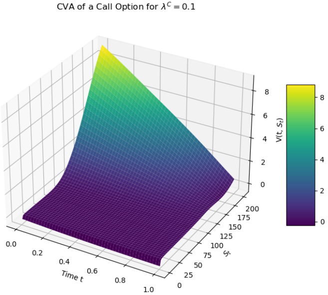



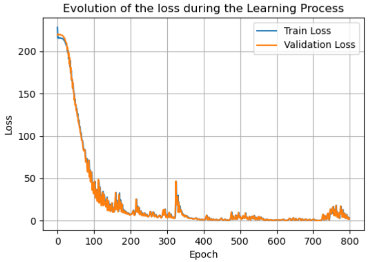

Figure: Evolution of the training and validation losses for the learning process of the PDE (6) related to the CVA price of a call option with $\lambda^C = 0.1$

### Remark

*We can see the terminal condition from the surface shape with $CVA(T, .) = 0$ and see that the function $S \to CVA(t, S)$ is an increasing function which is an expected behavior from the CVA definition.*

# Numerical results in the $B - S$ setting

## CVA on a Call Option with high $\lambda^C$



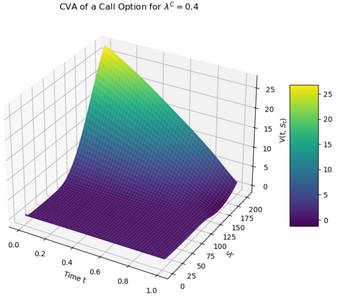




Figure: Evolution of the training and validation losses for the learning process of the PDE (6) related to the CVA price of a call option with $\lambda^C = 0.4$

### Remark

*We can see the overall value of CVA is higher in the case of $\lambda^C = 0.4$ than in the case of $\lambda^C = 0.1$ which just explains that the counterparty is more likely to default and so the CVA to be paid by the defaultable counterparty has to be higher.*

In a toy model (See [8] if you are interested in how we can obtain such system of PDE), we can show that $KVA$ and $FVA$ are solution to the following coupled systems of PDE associated respectively with $w$ and $v$ :

$$\frac{\partial v}{\partial t} + \mathcal{L}v + \lambda(\max(\alpha f \sigma S|\frac{\partial v}{\partial S} - \Delta_{bs}|, w) + v - u_{bs})^{-} - rv = 0 \quad (t,x) \in ]0,T[\times \mathbb{R}_*^+ \quad (7)$$

$$\frac{\partial w}{\partial t} + \mathcal{L}w + h\max(\alpha f \sigma S|\frac{\partial v}{\partial S} - \Delta_{bs}|, w) - (r+h)w = 0, \quad (t,x) \in ]0,T[\times \mathbb{R}_*^+ \quad (8)$$

$$v(T,S) = w(T,S) = 0 \quad x \in \mathbb{R}_*^+$$

where $h$ represents a dividend rate, $\alpha$ represents a mishedge parameter, $\lambda$ is a funding rate and $f$ is a quantile level. $u_{bs}$ and $\Delta_{bs}$ represent the call and delta price of a single call option of same characteristics as before.

## Remark

*In this couple PDE systems, we parametrize two neural networks $\mathcal{U}_1(\theta_1)$ and $\mathcal{U}_2(\theta_2)$ and we solve $\inf_{\theta=(\theta_1,\theta_2)} \mathbb{L}(\mathcal{U}(\theta))$ where $\mathbb{L}$ represents the operator associated to system of PDE (7) and (8). For the numerical experiments, we took $\alpha = 0.3$, $\lambda = 0.02$, $f = 1.2$ and $h = 0.1$.*

# Deep Galerkin for PDE

## System of coupled PDEs for KVA and FVA

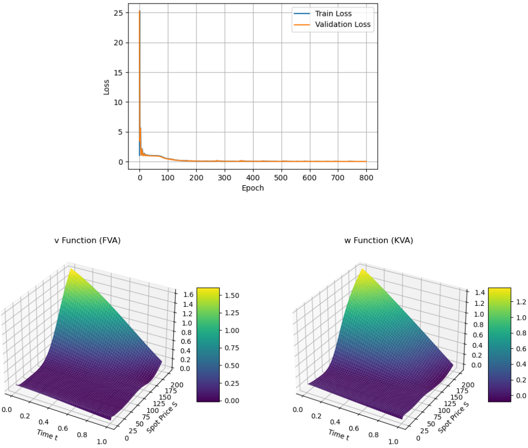



Figure: Evolution of the training and validation losses for the learning process of PDE for the coupled systems of PDE (7) and (8) for FVA and KVA and associated surfaces

# Contents

We are going to present some simple applications of the Deep BSDE Solver for solving PDE with the following form :

$$\partial_t v + \mathcal{L}v + f(x, v, \nabla_x v) = 0, \quad (t, x) \in [0, T) \times \mathbb{R}^d \qquad (9)$$
$$v(T, x) = g(x), \quad x \in \mathbb{R}^d$$

From this PDE, we can consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which supports a brownian motion $W = (W_t)_{t \geqslant 0}$ with his natural filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geqslant 0}$ and we can introduce the forward process $X = (X_t)_{t \geqslant 0}$ associated to the operator $\mathcal{L}$. Assuming this process known, we can consider the following pair of processes $(Y, Z)$ solving the following BSDE :

$$-dY_t = f(t, X_t, Y_t, Z_t) - Z_t dW_t, \quad 0 \leqslant t \leqslant T \qquad (10)$$
$$Y_T = g(X_T)$$

Applying Itô Formula to the process $v(t, X_t)$ with $v$ solving the PDE (9), we can see :

$$\begin{aligned} v(T, X_T) &= v(t, X_t) + \int_t^T (\partial_t v + \mathcal{L}[v])(s, X_s) ds + \int_t^T \nabla_x v(s, X_s)^\top dW_s \\ &= v(t, X_t) - \int_t^T f(s, v(s, X_s), \nabla_x v(s, X_s)) ds + \int_t^T \nabla_x v(s, X_s)^\top dW_s \end{aligned}$$

Differentiating this equation, we see that the process $v(t, X_t)$ solves the following BSDE :

$$\begin{aligned} -dv(t, X_t) &= f(t, v(t, X_t), \nabla_x v(t, X_t)) - \nabla_x v(t, X_t)^\top dW_t \\ v(T, X_T) &= g(X_T) \end{aligned}$$

From existence and unicity of the theory of BSDE under suitable assumptions on $f$ and $g$, we have the following representation for the pair $(Y, Z)$ :

$$Y_t = v(t, X_t) \quad dt \otimes d\mathbb{P} \quad a.e \tag{11}$$

$$Z_t = \nabla_x v(t, X_t) \quad dt \otimes d\mathbb{P} \quad a.e \tag{12}$$

The Deep BSDE Algorithm is based on the representation of the equations (11) and (12) as it means that founding $v$ is equivalent to founding $Y$. Therefore, going back to equation (10), we can look for discretization of $Y$ and approximating $v(t_i, X_{t_i})$ as $Y_{t_i}$. However, note that the scheme is backward in time which would need to approximation conditional expectations at each time. Therefore, the idea of the algorithm is to treat the process $Y$ as a forward process for an unknown $y_0$ and for process $Z$ . We can then define the following loss $\mathcal{L}$ for a given $y_0 \in \mathbb{R}$ and $\mathcal{Z}$ a squared adapted integrable process as :

$$\mathcal{L}(y_0, \mathcal{Z}) = \mathbb{E}[|Y_T^{y_0,\mathcal{Z}} - g(X_T)|^2] \tag{13}$$

Therefore, as $y_0$ and $\mathcal{Z}$ are unknown parameters, they can be learnt through neural networks assuming that $\mathcal{Z} = \mathcal{Z}(s, X_s)$. The idea is then to learn through a neural network the mapping $(t, x) \to \mathcal{Z}(t, x)$ using a neural network $\mathcal{Z}^\theta$ and to put $y_0$ as a trainable parameter of this neural network which will be learnt during the training process.

Therefore, the idea is to minimize the following loss error :

$$L(\theta) = \mathbb{E}[|Y_T^\theta - g(X_T)|^2] \tag{14}$$

where we set :

$$Y_t^\theta = y_0^\theta - \int_0^t f(X_s, Y_s^\theta, \mathcal{Z}^\theta(s, X_s))ds + \int_0^t \mathcal{Z}^\theta(s, X_s)^\top \sigma(s, X_s) dW_s \tag{15}$$

### Remark

*Of course, for the numerical experiments, we will discretize* (15) *using an Euler Scheme on a grid* $0 = t_0 < t_1 < \ldots < t_N = T$ *with time step* $\Delta t$ *: starting from* $Y_0^\theta = y_0^\theta$*, we have :*

$$Y_{t_{i+1}}^\theta = Y_{t_i}^\theta - f\big(X_{t_i}, Y_{t_i}^\theta, \mathcal{Z}_\theta(t_i, X_{t_i})^\top\big)\Delta t + \sigma(t_i, X_{t_i})\Delta W_{t_i}, \quad i = 0, \ldots, n-1$$

We now assume a $B - S$ model dynamics with the underlying dynamics $S = (S^1, \ldots, S^d)$ and $W = (W^1, \ldots, W^d)$ multidimensional brownian motion given by :

$$dS_t^i = S_t^i(rdt + \sigma^i dW_t^i), \quad S_0^i \in (\mathbb{R}_*^+), \quad i = 1, \ldots, d \tag{16}$$

Under the Option pricing theory in the $B - S$ model, we have the PDE (5) :

$$\partial_t C + \mathcal{L}C - rC = 0, \quad (t, x) \in [0, T) \times (\mathbb{R}_*^+)^d$$
$$C(T, x) = g(x), \quad x \in (\mathbb{R}_*^+)^d$$

where the infinitemisal generator is given by :

$$\mathcal{L}v(t,x) = b(t,x)^\top D_x v(t,x) + \frac{1}{2} Tr\big(\sigma(t,x)\sigma(t,x)^\top D_x^2 v(t,x)\big)$$

**Remark**

*Therefore, in this setting, the equivalent functions $f$ and $g$ are given by :*

- $f(t, x, y, z) = -ry$
- $g(x)$ *the option payoff*

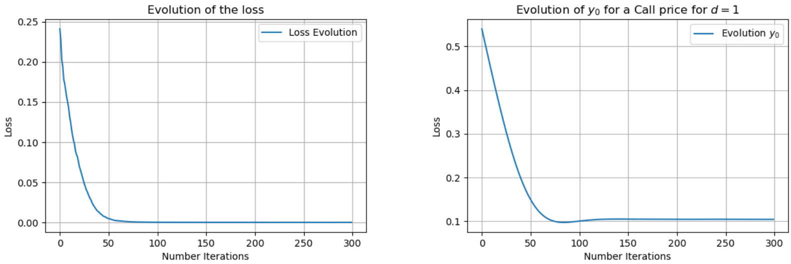
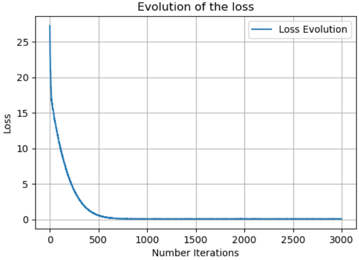



Figure: Loss and $y_0$ evolution for the $B - S$ PDE (5) for a Call option for $d = 1$ with payoff $g(x) = (x - K)^+$ with $r = 0.05$, $\sigma = 0.2$, $T = 1$, $x_0 = 1$ and $K = 1$.

Table: $u(0, x_0)$ Approximation for the Basket Call Option

| | True Value | Estimate Value |
|---|---|---|
| $u(0, x_0)$ | 0.1045 | 0.1043 |



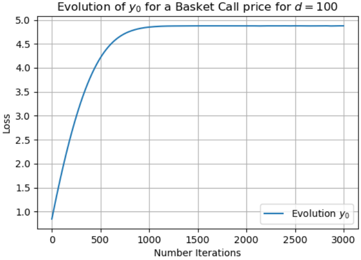




Figure: Loss and $y_0$ evolution for the $B - S$ PDE (5) for a Basket Call option for $d = 100$ with payoff $g(x) = (\sum_{i=1}^{d} x_i - dK)^+$ with $r = 0.05$ and $\sigma^i = 0.2$ for $i = 1, \dots, d$ and with uncorrelated $(W^i)_{i=1,\dots,d}$ with $x_0 = (1, \dots, 1) \in \mathbb{R}^d$.

Table: $u(0, x_0)$ Approximation for the Basket Call Option

| | Estimate Value |
|---|---|
| $u(0, x_0)$ | 4.8771 |

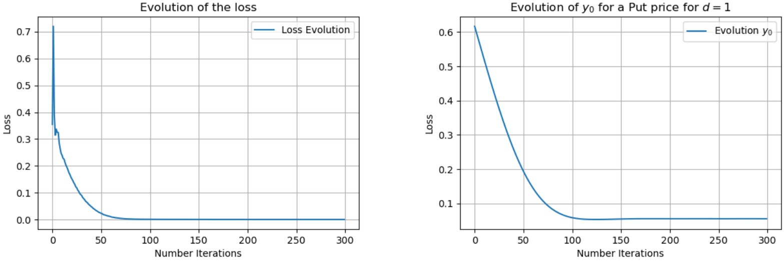



Figure: Loss and $y_0$ evolution for the $B - S$ PDE (5) for a Put option for $d = 1$ with payoff $g(x) = (K - x)^+$ with $r = 0.05$, $\sigma = 0.2$, $T = 1$, $x_0 = 1$ and $K = 1$.

Table: $u(0, x_0)$ Approximation for the Basket Put Option

| | True Value | Estimate Value |
|---|---|---|
| $u(0, x_0)$ | 0.05574 | 0.05568 |

# Numerical Results in the $B - S$ setting

## A Basket Put Option

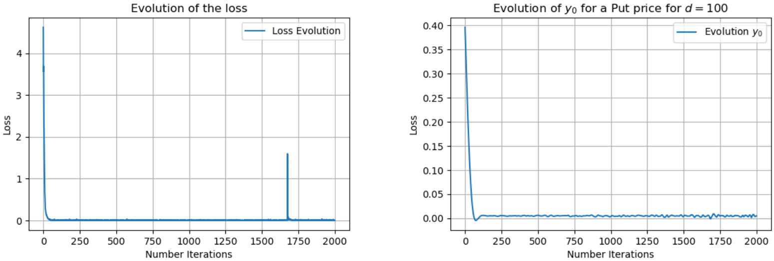



Figure: Loss and $y_0$ evolution for the $B - S$ PDE (5) for a Basket Put option for $d = 100$ with payoff $g(x) = (dK - \sum_{i=1}^{d} x_i)^+$ with $r = 0.05$ and $\sigma^i = 0.2 \quad i = 1, \ldots, d$ with uncorrelated $W = (W^1, \ldots, W^d)$ with $x_0 = (1, \ldots, 1) \in \mathbb{R}^d$.

Table: $u(0, x_0)$ Approximation for the Basket Put Option

| | Estimate Value |
|---|---|
| $u(0, x_0)$ | 0.0051 |

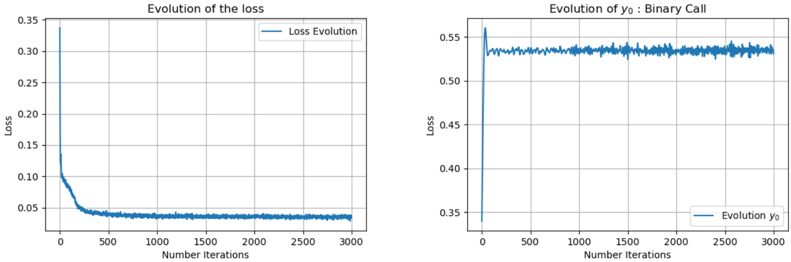



Figure: Loss and $y_0$ evolution for the $B - S$ PDE (5) for a Binary option for $d = 1$ with payoff $g(x) = \mathbb{1}_{x>K}$.

Table: $u(0, x_0)$ Approximation for the Binary Option

| | True Value | Estimate Value |
|---|---|---|
| $u(0, x_0)$ | 0.5323 | 0.5307 |

- Impact of the discontinuity of $g$ in the learning process.

The Allen-Cahn PDE is a famous *PDE* given by the following :

$$\partial_t v + \Delta_x v + v - v^3 = 0 \quad (t,x) \in [0, T( \times \mathbb{R}^d, \tag{17}$$
$$v(T,x) = \frac{1}{2 + \frac{2}{5}\|x\|^2} \quad x \in \mathbb{R}^d,$$

In this case, we can recover the *BSDE setting* with the following forward process :

$$dX_t = \sqrt{2} I_{d\times d} dW_t \in \mathbb{R}^d$$

and with the pair of process $(Y, Z)$ by setting :

- $f(t,x,y,z) = y - y^3$ with $y$ valued $\in \mathbb{R}$
- $g(x) = \frac{1}{2+\frac{2}{5}\|x\|^2} \in \mathbb{R}$

**Remark**

*In the numerical experiments, we will set* $T = \frac{3}{10}$ *with* $x_0 = 0$ *and* $d = 100$ *and try to recover the true estimate value of the PDE with such a setting like in the article [4].*

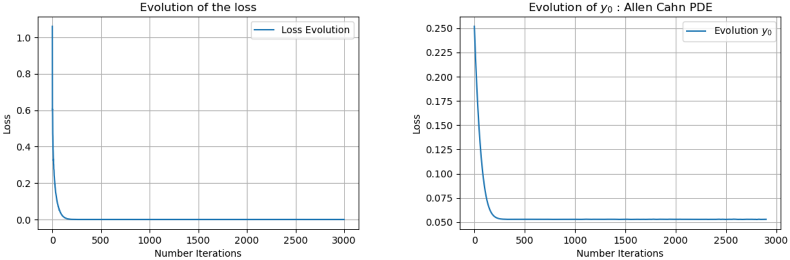



Figure: Loss and $y_0$ evolution for the Allen Cahn PDE (17)

Table: $u(0, x_0)$ Approximation for the Allen Cahn Equation

| | True Value | Estimate Value |
|---|---|---|
| $u(0, x_0)$ | 0.0528 | 0.0529 |

We consider the following PDE which can be shown to be the PDE arising from an HJB equation in optimal control :

$$\partial_t v + \Delta_x v - \frac{1}{2}|\nabla_x v|^2 = 0, \quad (t,x) \in [0,T) \times \mathbb{R}^d, \tag{18}$$
$$v(T,x) = g(x), \quad x \in \mathbb{R}^d,$$

In this case, we can recover the *BSDE setting* with the following forward process :

$$dX_t = \sqrt{2} I_{d \times d} dW_t \in \mathbb{R}^d$$

and with the pair of process $(Y, Z)$ by setting :

- $f(t,x,y,z) = -\|z\|^2$ with $z \in \mathbb{R}^{1 \times d}$

**Remark**

*For the numerical experiments, we choose $x_0 = 0$, $d = 100$, and $g(x) = ln(\frac{1}{2}(1 + \|x\|^2)$ with semi-explicit form given by Hopf-Cole transformation for benchmark value :*

$$v(0,x_0) = -ln\Big(\mathbb{E}\Big[exp\big(-g(x_0 + \sigma W_T)\big)\Big]\Big)$$

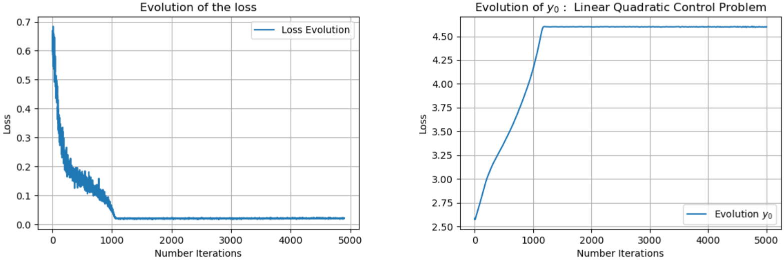



Figure: Loss and $y_0$ evolution for the Linear Quadratic control problem from PDE (18)

Table: $y_0 = u(0, x_0)$ Approximation for the Linear Quadratic control problem

| | True Value | Estimate Value |
|---|---|---|
| $u(0, x_0)$ | 4.5901 | 4.5988 |

**Pros :**

- Probabilistic representation is helpful for the choice of training samples, convergence analysis.
- Very efficient in very high dimension $d >> 1$.
- Easy implementation of neural networks through Python packages like PyTorch or Tensorflow. See the Python notebook.

**Cons :**

- Can be unstable with a large number of timesteps $N$.
- Can only be used for semi-linear PDEs through their probabilistic representation.

**Sum up of the presentation :**

- Deep learning methods provide a breakthrough for the computation challenge of solving high dimensional nonlinear problem arising in PDEs and stochastic control.
- Easy implementation of neural networks with Python packages.
- Deep Galerkin can "always" be used when PDE too complex with no probabilistic representation.

**To Go Further on Deep Learning methods for PDE and MDP :**

- What about algorithms for solving American Options in High Dimension which are related to variation inequalities ?See the *BDP* algorithm in [6].
- Extension of Deep BSDE Solver for Jump Processes : See [3].
- Other applications of Deep Learning : See [1] and [2] for applications of Deep Learning for solving Markov decision processes (*MDP*).

# References


- C.Huré, H.Pham, A.Bachouch, N.Langrené, 2019, *"Deep Neural Networks algorithms for stochastic control problems on finite horizon : Convergence Analysis"*
- A.Bachouch, C.Huré, N.Langrené, H.Pham, 2019, *"Deep neural networks algorithms for stochastic control problems on finite horizon: numerical applications"*
- K.Andersson, A.Gnoatto, M.Patacca, A.Picarelli, 2024, *"A Deep BSDE solver with jumps"*
- Weinan.E, J.Han, A.Jentzen, 2017, *"Deep learning-based numerical methods for high-dimensional parabolic partial differential equations and backward stochastic differential equations"*
- A.Gnoatto, A.Picarelli, C.Reinsiger, 2022, *"Deep XVA solver - A neural network based counterparty credit risk management framework"*
- M.Germain, H.Pham, X.Warin, 2021, *"Neural networks-based algorithms for stochastic control and PDEs in finance"*
- J.Sirignano, K.Spiliopoulos, 2017 *"DGM: A deep learning algorithm for solving partial differential equations"*
- Y.Armenti, S.Crépey, C.Zhou, 2018, *"The Sustainable Black-Scholes Equations"*